Finite Element Modeling of Frequency-Dependent Material Damping Using Augmenting Thermodynamic Fields

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A new method of modeling frequency-dependent material damping in structural dynamics analysis is reported. Motivated by results from materials science, augmenting thermodynamic fields are introduced to interact with the usual mechanical displacement field. The methods of irreversible thermodynamics are used to develop coupled material constitutive relations and partial differential equations of evolution. These equations are implemented for numerical solution within the computational framework of the finite-element method. The method is illustrated using several examples including longitudinal vibration of a rod, transverse vibration of a beam, and vibration of a large space truss structure.

Nomenclature

A = ATF first-order element or system matrix ("augmented mass matrix")
B = ATF zeroth-order element of system matrix ("augmented stiffness matrix")
$f(x)$ = ATF for beam bending, corresponding to $\xi(x)$
$\boldsymbol{p}$ = vector of discrete ATF displacements
$\boldsymbol{q}$ = vector of discrete mechanical displacements
$u(x)$ = mechanical displacement field, parallel to x in lineal elements
$v(x)$ = mechanical displacement field, transverse to x for lineal elements
x = independent variable for lineal structural elements
$\boldsymbol{x}$ = vector of element or global degrees of freedom
$\epsilon(x)$ = strain field
$\gamma(x)$ = ATF, gradient of $\xi(x)$
λ = complex eigenvalue
$\sigma(x)$ = stress field
$\sigma^*(x)$ = measure of irreversibility related to entropy generation rate
ω = radian frequency
$\xi(x)$ = augmenting thermodynamic field (ATF)
ζ = damping ratio

Introduction

Importance of Material Damping

In advanced engineering systems such as large space structures (LSS) or robots, the combination of severe disturbances, stringent requirements, and structural design constraints can result in structures that exhibit significant flexibility. Passive and active damping of these structures is important for several reasons. In terms of performance, higher damping can reduce steady-state vibration levels and can reduce the time needed for transient vibrations to settle. Inherent passive damping can reduce the magnitude of control needed and can reduce control system complexity. Passive damping can also strongly couple vibration modes that are closely spaced in frequency and computed assuming no damping. Most important, however, the design of stable, fast-responding control systems benefits from accurate knowledge of structural dynamic behavior, which depends on the magnitudes and mechanisms of inherent damping.

Material damping is likely to be an important, perhaps dominant, contributor to damping in "monolithic" structures and to on-orbit damping in precision spacecraft. In common built-up structures that operate in the atmosphere, air damping and joint damping typically dominate system damping. However, air damping is clearly eliminated in space, and the effects of joint damping will be reduced because of requirements for precision ("tight" joints) and typically low vibration levels (friction "lockup").

Material damping is generally a complex function of frequency, temperature, type of deformation, amplitude, and structural geometry. Figure 1, adapted from the frontispiece of the pioneering text *Elasticity and Anelasticity of Metals*,[1] illustrates the typical frequency dependence of material damping. Current popular treatments of damping in structural dynamics are not physically motivated and are unable to reproduce this fundamental behavior.

Current Damping Models in Structural Dynamics

Perhaps the most common equation in linear computational structural dynamics is

$$M\ddot{x} + C\dot{x} + Kx = f \tag{1}$$

The properties of M (the "mass matrix") and K (the "stiffness matrix") are well known and are not discussed here. The "viscous damping matrix," C, can be shown to be positive semidefinite and symmetric, as is K. However, Eq. (1) has a fundamental flaw: based on viscous damping, it has the wrong structure to describe the actual microscopic mechanisms responsible for material damping—no such element-based damping matrix accurately represents the behavior of real engineering materials over a broad frequency range.

Several methods for incorporating material damping into structural models have been used, and continue to be used within the engineering community. These methods include viscous damping, frequency-dependent viscous damping, complex modulus, hysteretic damping, structural damping, vis-

Received Feb. 2, 1989; presented as Paper 89-1380 at the AIAA 30th Structures, Structural Dynamics, and Materials Conference, Mobile, AL, April 3-5, 1989; revision received July 17, 1989.

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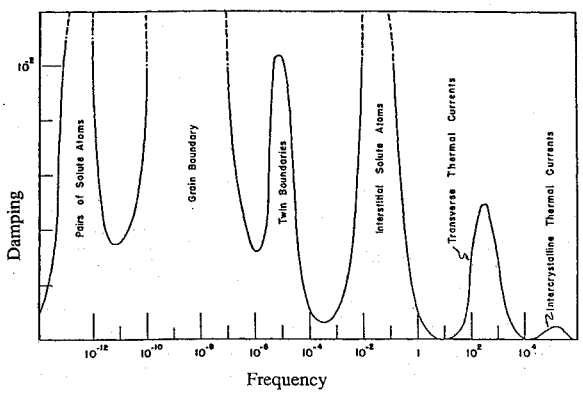

Fig. 1 Typical frequency-dependence of material damping.

coelasticity, hereditary integrals, and modal damping.[2,3] Each has some utility, but each suffers from one flaw or another. Although some of these (e.g., viscoelasticity) have the potential for better accuracy than more widely used methods, they are not commonly used in the engineering community—perhaps because of the lack of physical motivation or difficulty of use.

Inherent material damping is an important factor in the design of LSS, robots, and other advanced engineering systems, but existing damping models are inadequate. This paper reports the initial results of a research effort undertaken to address this fundamental problem.

Related Research

This paper presents a time-domain continuum model of material damping that exhibits the characteristic frequency dependence of real materials—a physically motivated model compatible with current finite element structural analysis methods. The approach involves the introduction of augmenting thermodynamic fields (ATF). Several results in the recent engineering literature are closely related to the subject work and are briefly discussed below.

Golla, Hughes, and McTavish (GHM) set out to derive a time-domain finite element formulation of viscoelastic material damping.[4,5] Their work was motivated by the same general perceived need as this work, but was guided by the observation that experimental results, often recorded in the frequency domain, are of little direct use in time-domain models.

The results reported here resemble theirs in some respects, for example, in the introduction of additional "dissipation coordinates." However, the results differ in important ways: First, their approach is fundamentally a mathematical one, deriving time-domain realizations from frequency-domain models—no attempt is made to provide a physical interpretation of the dissipation coordinates as thermodynamic field variables with a direct relationship to microstructural features of real materials. Second, their model is restricted to consideration of what is termed here "microstructural damping." Third, they restrict the mathematical form of the augmenting equations to be second order, the same as that of the fundamental equations of mechanical vibration—this seems unnecessary. As shown in their examples, however, the GHM technique can be used successfully to fit a portion of an experimentally determined curve of damping vs frequency, and standard analysis tools can be used to solve the resulting equations.

Both ATF and GHM have advantages over the more conventional modal strain energy (MSE) modeling method in that they are time-domain models; modal damping is calculated concurrently with modal frequency (no lookup tables or iterative procedures are required to converge on both), and the resulting complex modes more accurately reflect the relative phase of vibration at various points on a structure. The ATF method is primarily distinguished from the GHM approach in that it is a *direct* time-domain formulation amenable to numerical treatment using conventional finite element methods. Like GHM, ATF employs additional coordinates to more accurately model damping (nearly pure thermodynamic relaxation modes are present in response spectra when using either), but the physical interpretation of these coordinates is more obvious in the ATF approach. Finally, the dissipation coordinates of GHM are internal to individual elements, whereas the augmenting thermodynamic fields of ATF are continuous from element to element.

Segalman has recently addressed the calculation of stiffness and damping matrices for structures made from linearly viscoelastic materials.[6] His is essentially a perturbation technique: the perturbation solution for a "slightly viscoelastic" structure is required to match the corresponding solution for a "slightly viscously damped" structure. He works exclusively in the time domain and avoids introducing additional coordinates. However, the resulting stiffness and damping matrices are generally unsymmetric, and the assumption of "small viscoelasticity" may limit the utility of the approach.

Torvik and Bagley have also developed a relevant model of material damping.[7] The core of their concept is the use of fractional time derivatives in material constitutive equations. Their development was motivated by the observation that the frequency dependence observed in real materials is often weaker than the dependence predicted by first-order viscoelastic models. With four- and five-parameter models, they have been able to accurately represent the elastic and dissipative behavior of over 100 materials over frequency ranges as broad as eight decades. For most viscoelastic polymeric materials they have examined, the parameter representing the order of differentiation is in the range one-half to two-thirds. The application of the general fractional derivative approach to time-domain analysis, however, is cumbersome and is an area of continuing research.

Other relevant, current work in the engineering aspects of material damping focuses primarily on the development of experimental techniques and measurement of damping in various materials.[8-10] In addition, the use of the MSE method for estimating the damping of built-up structures and composite materials from the measured damping of constituents continues to grow.[11-14]

Background and Approach

Material Damping

In no engineering material is the strain a function of stress alone. This "anelasticity" means that, in response to a change in the imposed mechanical conditions, time is required for the material to reach thermodynamic equilibrium. The tendency of a thermodynamic system to evolve toward an equilibrium state is termed "relaxation." When the equilibrium state depends on the value of a mechanical variable (a stress or a strain), the phenomenon is known as "anelastic relaxation." Anelastic relaxation is inherently a thermodynamic phenomenon that arises as the result of the mutual coupling of stress and strain to other thermodynamical variables—variables that can change to new equilibrium values only through time-dependent kinetic processes such as diffusion.[1,15]

Structural dynamicists are the unintended beneficiaries of a sizable literature on material damping.[16] For many years, crystallographers and metallurgists have used "internal friction" as a probe into the underlying structure of materials. By measuring damping as a function of frequency, temperature, load type, and amplitude, they can determine the mobility and activation energies of various microstructural features of materials. These researchers have identified a multitude of internal variables and relaxation mechanisms that range in geometrical scale from crystal lattice dimensions on up to structural dimensions. As previously noted, material damping is generally a

complex function of frequency, temperature, load type, amplitude of vibration, and sometimes structural geometry.

Irreversible Thermodynamics

It is an oversimplification to state that dynamics is the study of the evolution of momentum, but it provides useful contrast to the statement that thermodynamics is the study of the evolution of energy. Material damping, involving the transfer of energy from structural vibration to other forms of energy, is fundamentally a thermodynamic phenomenon. Analysis of material damping should rest on a thermodynamical foundation.

The field of nonequilibrium thermodynamics provides a general framework for the macroscopic description of irreversible processes.[17] In this field, the so-called balance equation for the entropy plays a central role. This equation expresses the fact that the entropy of a volume element changes with time for two reasons. First, it changes with time because entropy flows into the volume element and second, because there is an entropy source due to irreversible phenomena inside the volume element. The entropy source is always a non-negative quantity, since entropy can only be created, never destroyed. For reversible transformations, the entropy source vanishes. This is the local formulation of the second law of thermodynamics.

The entropy source term often has a very simple appearance: it is a sum of terms each being a product of a flux characterizing an irreversible relaxation process, and a quantity called thermodynamic force, which is related to the nonuniformity of the system (the gradient of the temperature, for instance) or to the deviations of some internal state variables from their equilibrium values (the affinity, for instance). The entropy source strength can thus serve as a basis for the systematic description of the irreversible processes occurring in a system.

Finite Element Method

The finite element method is arguably the most powerful and popular method for solving field equations of evolution. Piecewise-continuous trial displacement functions are first assumed over a local region of the system being analyzed. Individual element matrices are then computed using the method of weighted residuals (MWR) or variational principles and assembled into global system matrices. The output of this procedure is a set of discretized equations of motion that are readily solved using standard computational techniques on digital computers.

Approach

The physically significant "internal state variables" of materials science play a central role in this work, motivating the introduction of ATF to interact with the usual mechanical displacement field of continuum structural dynamics. The techniques of nonequilibrium, irreversible thermodynamics are used to develop coupled material constitutive equations and coupled partial differential equations of evolution. Constitutive equations of damped materials describe the coupling of all dependent fields (e.g., as the coefficient of thermal expansion couples the displacement and temperature fields in thermoelasticity).

Field equations have been derived for the general three-dimensional case,[18] but space limitations preclude discussion here. To illustrate the ATF modeling method, the general field equations are specialized to the simplest continuum case, viz., that of one-dimensional vibration of an isotropic rod. For simplicity, a single augmenting field is employed. In practice, however, additional fields could be used as needed to better approximate experimental data over a frequency range of interest. An alternate form of the governing equations is also investigated and, although thermoelastic damping has been considered,[18] it is not discussed in detail here.

The solution of the coupled partial differential equations is addressed in several ways. Analytical Fourier analysis yields an approximate relationship between damping and frequency, whereas numerical finite element analysis results in a set of coupled discrete differential equations of evolution. A free vibration eigenvalue problem for these discrete equations can then be defined and solved to yield complex modes.

MWR is used to develop damped finite element matrices. Because it can reduce the order and continuity required of assumed approximate displacement fields, integration by parts is an important part of the process of developing element matrices in MWR. However, its use can also present an analyst with a choice between alternate matrices with no a priori rules for choosing between them. We give here the results of a series of numerical experiments that were performed for the ATF-damped rod elements to identify those matrices superior in terms of convergence to the solution obtained via Fourier analysis.

ATF-damped finite elements for planar bending beams are also developed and demonstrated with a modal analysis of a cantilevered beam. A key assumption concerns the variation of the augmenting field through the thickness of the beam—this assumption is consistent with the Bernoulli-Euler theory.

Finally, the utility of the ATF modeling method is demonstrated in a modal analysis of a 10-bay, 30-m planar truss structure. Such a structure resembles those proposed for many future space missions. The damped rod elements already generated are modified to include the kinetic energy of transverse motion, and interelement continuity of the augmenting fields is addressed. The ATF method readily accommodates the use of multiple materials, and ATF-damped elements can be used concurrently with undamped elastic elements.

Longitudinal Vibration of a Rod

Governing Equations

Consider the case of one-dimensional motion corresponding to longitudinal vibration of a thin rod. The mechanical displacement along the rod is denoted by $u(x)$ [strain $\epsilon(x) = u'(x)$], and the rod has uniform mass density ρ and unrelaxed or dynamic modulus of elasticity E. A single augmenting thermodynamic field $\xi(x)$ is introduced. The fields that are thermodynamically conjugate to ϵ and ξ are the stress σ and the affinity A. The affinity can be interpreted as a thermodynamic "force" driving ξ toward equilibrium. The material property δ describes the strength of the coupling of the two dependent fields, u and ξ. In the absence of coupling of the two fields, increments of stress and strain are proportional, with E the relating factor. Analogously, α is the material property that relates changes in A to those in ξ. To derive constitutive equations, the Helmholtz free energy density f, a thermodynamic potential appropriate for use when strain is an independent variable, is employed:

$$f = \tfrac{1}{2}E\epsilon^2 - \delta\epsilon\xi + \tfrac{1}{2}\alpha\xi^2$$

The material constitutive equations may be found as

$$\sigma = \frac{\partial f}{\partial \epsilon} = E\epsilon - \delta\xi$$

$$A = -\frac{\partial f}{\partial \xi} = \delta\epsilon - \alpha\xi$$

The usual stress-strain constitutive relations are seen to be augmented by an additional term in ξ, similar to the way in which temperature changes couple to stress and strain. The equation of evolution for the mechanical displacement field is developed from consideration of momentum balance (zero body forces are assumed). The equation of evolution for the augmenting thermodynamic field ξ is found through the use of a basic assumption of irreversible thermodynamics, namely that the rate of change of ξ is proportional to A, or in other

words, that the rate of change of ξ is proportional to its deviation from an equilibrium value. This results in a first-order differential equation, a "relaxation" equation:

$$\dot{\xi} = LA = -B(\xi - \bar{\xi})$$

where the equilibrium value of ξ is just that at which $A = 0$:

$$\bar{\xi} \equiv \xi\,|_{A=0} = (\delta/\alpha)\epsilon$$

The result is a set of two coupled partial differential equations in u and ξ:

$$\rho\ddot{u} - Eu'' = -\delta\xi'$$

$$\dot{\xi} + B\xi = (B\delta/\alpha)u'$$

A bilinear variational principle that generates these equations has also been developed,[18] and it leads to insight concerning the boundary conditions on the displacement and augmenting thermodynamic fields. The augmenting thermodynamic field is essentially an *internal* field, i.e, there are no explicit boundary conditions that it alone must satisfy. However, the mechanical displacement field must satisfy either displacement ("geometric") or stress ("natural") boundary conditions at each end of the rod, as is the case in undamped structural dynamics. Note that the stress boundary condition involves the augmenting field ξ:

$$\sigma(x) = Eu'(x) - \delta\xi(x) \quad \text{at} \quad x = 0 \quad \text{and/or at} \quad x = L$$

Under appropriate conditions, the governing equations can be shown to be dissipative and well-posed, guaranteeing a solution that depends continuously on the initial conditions.[18] Fourier analysis of these linear equations reveals that the damping and effective modulus are frequency dependent. An approximate equation for the damping ratio (for small damping) is[18]

$$\zeta = \frac{1}{4}\left(\frac{\delta^2}{E\alpha}\right)\frac{2(\omega/B)}{(1+(\omega/B)^2)}$$

This result is in accord with experimental results obtained by materials scientists for many microstructural damping mechanisms (including thermoelastic damping), although, as noted by Torvik and Bagley,[7] many materials exhibit weaker frequency dependence. Note that peak damping occurs at $\omega = B$, and that the magnitude of the peak depends on the strength of the coupling of the two equations. Also note that, in accord with the principles of irreversible thermodynamics, the entropy generation rate may be expressed as

$$\sigma^* = (1/T)(\alpha/\beta)\dot{\xi}^2$$

Finally, note that multiple augmenting fields could be used to better approximate experimental data over a given frequency range (e.g., by curve fitting as in the GHM method[4,5])—a single one is employed here for clarity of presentation.

An alternative formulation of this one-dimensional case may be considered. For example, the preceding equations can be expressed in terms of γ, the gradient of the ξ field, as follows:

$$\rho\ddot{u} - Eu'' = -\delta\gamma \tag{2a}$$

$$\dot{\gamma} + B\gamma = (B\delta/\alpha)u'' \tag{2b}$$

This formulation contains only even spatial derivatives and leads to some benefits in numerical solution, such as symmetric element submatrices.

A similar procedure can be used to derive the (linearized) one-dimensional equations of dynamic thermoelasticity, but is not shown here. However, a significant difference is that thermal relaxation depends on spatial derivatives of the temperature field, whereas relaxation of the previously treated augmenting thermodynamic fields depends only on local deviations from an equilibrium value. Damping due to such a scalar relaxation process is termed "microstructural damping."

Finite Element Treatment

The method of weighted residuals is used to develop element matrices. The u-γ formulation of the equations has been shown to be superior to the u-ξ formulation[18] and is employed here. Integration by parts is employed, changing the continuity required of the approximating and weighting functions.

Weighting Eq. (2a) by $b(x)$, Eq. (2b) by $\beta(x)$, and integrating by parts over the entire region of interests, yields the following weighted, coupled *PDE* (plus boundary terms):

$$b\rho\ddot{u} + b'Eu' = -b\delta\gamma \tag{3a}$$

$$\beta\dot{\gamma} + \beta B\gamma = -\beta'(B\delta/\alpha)u' \tag{3b}$$

The derivation for a single lineal element of length L is illustrated below. The same functions used to approximate the behavior of the dependent fields in the spatial region bounded by the element are used here as weighting functions—when there is only one dependent field, this is known as Galerkin's method. Let $u(x)$ and $b(x)$ be approximated by

$$u(x) = \Phi^T(x)Cq$$

where q is the vector of nodal mechanical displacements. Employ a similar approximation for $\gamma(x)$ and $\beta(x)$:

$$\gamma(x) = \Theta^T(x)Dp$$

where p is the vector of nodal ATF displacements.

Substituting the preceding equations for u and γ into Eqs. (3) and integrating over the length of the element, one finds

$$\int_0^L \{\rho C^T\Phi\Phi^T C\ddot{q} + EC^T\Phi'\Phi'^T Cq\}\,dx = \int_0^L \{-\delta C^T\Phi\Theta^T Dp\}\,dx$$

$$\int_0^L \{D^T\Theta\Theta^T D\dot{p} + BD^T\Theta\Theta^T Dp\}\,dx$$

$$= \int_0^L \left\{-\left(\frac{B\delta}{\alpha}\right)D^T\Theta'\Phi'^T Cq\right\}dx$$

These sets of equations may be written as

$$M\ddot{q} + Kq = -Bp$$

$$C\dot{p} + Hp = -Fq$$

or, in first-order form, as

$$\begin{bmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & C \end{bmatrix}\begin{Bmatrix} \ddot{q} \\ \dot{q} \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} 0 & K & B \\ -I & 0 & 0 \\ 0 & F & H \end{bmatrix}\begin{Bmatrix} \dot{q} \\ q \\ p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4}$$

In this treatment, both fields are approximated with linear interpolation functions using

$$\Phi^T = \begin{vmatrix} 1 & x \end{vmatrix}$$

Figure 2 illustrates the element and the nodal values for the two dependent fields, u and γ.

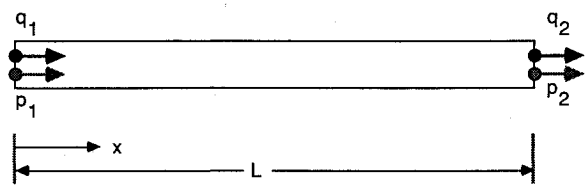

Fig. 2 An ATF-damped rod element.

If the elemental degrees of freedom are ordered to facilitate assembly as

$$x = [\dot{q}_1 \quad q_1 \quad p_1 \quad \dot{q}_2 \quad q_2 \quad p_2]^T$$

the elemental equations may be expressed as

$$A\dot{x} + Bx = 0$$

and the ATF-damped rod element matrices are

$$A = \begin{bmatrix} \left(\frac{\rho AL}{3}\right) & 0 & 0 & \left(\frac{\rho AL}{6}\right) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{AL}{3}\right) & 0 & 0 & \left(\frac{AL}{6}\right) \\ \left(\frac{\rho AL}{6}\right) & 0 & 0 & \left(\frac{\rho AL}{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \left(\frac{AL}{6}\right) & 0 & 0 & \left(\frac{AL}{3}\right) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \left(\frac{EA}{L}\right) & \left(\frac{\delta AL}{3}\right) & 0 & \left(-\frac{EA}{L}\right) & \left(\frac{\delta AL}{6}\right) \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{B\delta A}{\alpha L}\right) & \left(\frac{BAL}{3}\right) & 0 & \left(-\frac{B\delta A}{\alpha L}\right) & \left(\frac{BAL}{6}\right) \\ 0 & \left(-\frac{EA}{L}\right) & \left(\frac{\delta AL}{6}\right) & 0 & \left(\frac{EA}{L}\right) & \left(\frac{\delta AL}{3}\right) \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \left(-\frac{B\delta A}{\alpha L}\right) & \left(\frac{BAL}{6}\right) & 0 & \left(\frac{B\delta A}{\alpha L}\right) & \left(\frac{BAL}{3}\right) \end{bmatrix}$$

Note that E is the unrelaxed, or high-frequency asymptotic modulus of the material. Structural dynamicists more often use the relaxed, or static modulus in analysis.

Numerical Results for Free Vibration Eigenvalue Problem

To evaluate the performance of this formulation of an ATF-damped rod element, a specific problem is addressed, namely the determination of the natural modes of vibration of a free-free rod. Accordingly, no geometric boundary conditions are enforced. The results are compared to those of approximate Fourier analysis.

Assuming a solution for $x(t)$ in the form $e^{\lambda t}$, the following eigenvalue problem is defined:

$$[\lambda A + B]x = 0$$

The matrix equations of motion are formulated, and this problem is solved to yield complex eigenvalues λ and mode shapes x. Element matrices may be assembled into global system matrices in the usual manner of structural finite-element analysis. The damping ratio for each mode is calculated as the ratio of the negative of the real part of the eigenvalue to the total magnitude. The damping ratio ζ is then plotted against the magnitude of the eigenvalue. Note that the spectrum of eigenvalues will contain vibration modes, relaxation modes, and rigid-body modes. In the complex plane, the damped vibration modes lie near the imaginary axis, slightly in the LHP with negative real parts; the relaxation modes lie on the negative real axis. These relaxation modes are characteristic of the response of the γ field.

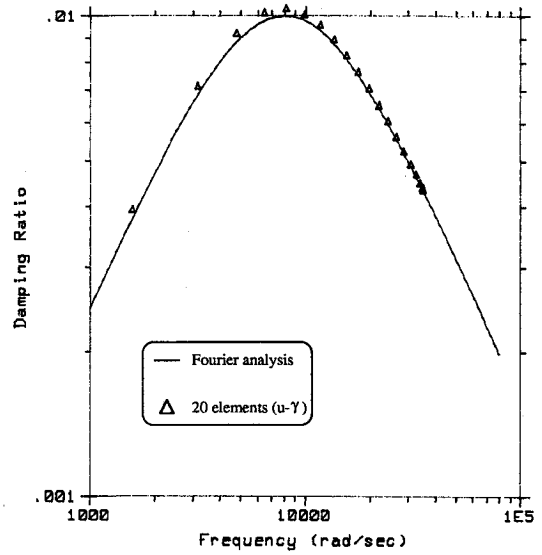

Fig. 3 Frequency-dependent damping exhibited in modal analysis of ATF-damped free-free rod.

The numerical parameter values used are

$E = 7.13e10$
$\rho = 2750$
$L = 10$
$B = 8000$
$\alpha = 8000$
$\delta = 4.7766e6$

The elastic properties correspond roughly to those of aluminum in International System units. Using the results of the Fourier analysis, these values were chosen to yield a peak damping ratio of 0.01 at the frequency of the 5th mode. Note that numerical values for α and δ cannot be uniquely specified.

Figure 3 shows typical numerical results obtained using this approach with 20 damped rod elements. The solid line shows the results of the approximate Fourier analysis, while the triangular symbols show the results of the damped modal analysis. The characteristic variation of material damping with frequency is apparent in both the Fourier analysis results and the finite element results and, as previously noted, conventional damping modeling techniques are incapable of producing such results. The frequencies predicted using this method appear to converge from above, as is the case with undamped elements. The results of this alternate u-γ formulation agree quite well at all frequencies with those expected on the basis of the Fourier analysis and are superior to those of the basic u-ξ formulation.[18]

In light of these results, the earlier statement that Eq. (1) was fundamentally flawed is now more clear. Equation (4) has a structure that is more complex than that of Eq. (1), but it is what is required to accurately model many kinds of material damping. Because of its apparent good performance, the ATF-damped rod elements developed in this section will be used as the basis for a truss element in a later section.

Transverse Vibration of a Beam

In this section, the PDEs governing the motion of a planar Bernoulli-Euler beam with ATF damping are derived. A single augmenting field is considered for convenience. Damped finite elements for beam bending are generated, and a corresponding free-vibration eigenvalue problem is solved to investigate the frequency dependence of damping and to determine the performance of the elements.

Governing Equations

The derivation begins by recalling some elements of engineering beam theory. A uniform, planar, slender beam made of an isotropic material is considered. When the beam is deformed, transverse sections that were originally plane remain so. The stress normal to a transverse plane (parallel to the beam axis) is assumed to be the only stress of significance.

The x axis is defined to be parallel to the beam axis with a corresponding vector displacement component $u(x,y,z)$. The y axis is defined to be normal to the x axis, with corresponding displacement $v(x,y,z)$. The following equations express the fundamental approximations of this theory:

$$v(x,y,z) = v(x)$$

$$u(x,y,z) = -yv'(x)$$

With no distributed external loads, the fundamental equation is

$$\rho A\ddot{v} = \frac{\partial^2}{\partial x^2}\int_A \sigma_{xx} y \, dA$$

For a uniform beam, the spatial derivatives may be moved inside the integral. The relevant material constitutive equation is that previously derived for one-dimensional rod motion:

$$\sigma_{xx} = Eu' - \delta\xi$$

Using the definition

$$I = \int_A y^2 \, dA$$

for the second moment of area referred to neutral axis (at $y = 0$) and substituting the constitutive equation into the fundamental PDE, the following equation results:

$$\rho A\ddot{v} + EIv^{iv} = -\delta\int_A \xi''(x,y) y \, dA$$

In addition, the previously found equation for the evolution of $\xi(x,y)$ is integrated over the beam cross section to yield

$$\int_A \{\dot{\xi} + B\xi\} y \, dA = -I\left(\frac{\delta B}{\alpha}\right) v''$$

Now, a critical assumption is made, namely that $\xi(x,y)$ varies linearly with y across the cross section:

$$\xi(x,y) = yf(x)$$

Indeed, if the case of static deformation with relaxation of ξ precluded is considered, this is precisely the situation that results. This leads to considerable simplification:

$$\int_A \xi'' y \, dA = If''$$

and the set of coupled equations

$$\rho A\ddot{v} + EIv^{iv} = -\delta If''$$

$$\dot{f} + Bf = -\left(\frac{\delta B}{\alpha}\right) v''$$

As before, boundary conditions may be specified for the mechanical displacement and the stress, but not for the augmenting field. Two quantities must be specified at each end of the beam: either the displacement or the shear force and the rotation or the bending moment. Note that the stress boundary conditions (moment and shear) again involve the augmenting field.

Finite Element Treatment

The finite elements derived here build upon the "standard planar bending element," which employs a cubic approximation for $v(x)$ and has both slope and displacement continuity between elements. A linear approximation is employed for $f(x)$. The derivation for a single element is illustrated in what follows. Figure 4 illustrates the element and the nodal values for the two dependent fields v and f. Let $v(x)$ be approximated by

$$v(x) = \Phi^T(x)Cq$$

where

$$\Phi^T = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix}$$

and let $f(x)$ be approximated by

$$f(x) = \Theta^T(x)Dp$$

where

$$\Theta^T = \begin{bmatrix} 1 & x \end{bmatrix}$$

Using a procedure similar to that employed in the previous section, one finds again that alternate element matrices may be derived depending upon how integration by parts is employed. The only a priori reason to choose one from the alternatives is on the basis of an examination of the resulting boundary terms. For this problem, numerical experiments were performed to identify the best in terms of agreement with the Fourier analysis results.[18] The following proved to yield superior performance as well as the expected boundary terms:

$$\int_0^L \{\rho AC^T\Phi\Phi^T C\ddot{q} + EIC^T\Phi''\Phi''^T Cq\} \, dx = \int_0^L \{-\delta IC^T\Phi''\Theta^T Dp\} \, dx$$

$$\int_0^L \{D^T\Theta\Theta^T\dot{p} + BD^T\Theta\Theta^T Dp\} \, dx = \int_0^L \left\{-\left(\frac{\delta B}{\alpha}\right) D^T\Theta\Phi''^T Cq\right\} dx$$

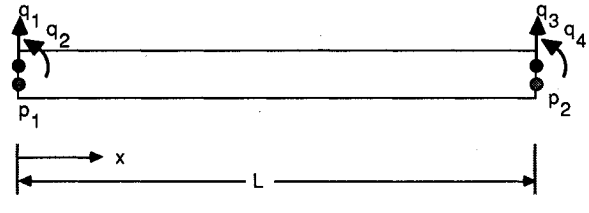

Fig. 4 An ATF-damped planar beam bending element.

Carrying out the integration, these may be written as

$$M\ddot{q} + Kq = B_2 p$$

$$C\dot{p} + Hp = F_2 q$$

Expressed in first-order form, the governing matrix differential equations have a structure similar to those of the damped rod elements:

$$\begin{bmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & C \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \dot{q} \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} 0 & K & B \\ -I & 0 & 0 \\ 0 & F & H \end{bmatrix} \begin{Bmatrix} \dot{q} \\ q \\ p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

In these equations, the M and K submatrices are the usual mass and stiffness matrices for uniform planar bending elements (based on a cubic approximation function); M is positive definite and symmetric, and K is positive semidefinite and symmetric. C and H are also positive definite and symmetric, like M, but of lower dimension.

The best coupling submatrices B and F are

$$B_2 = \frac{\delta I}{L} \begin{bmatrix} -1 & 1 \\ -L & 0 \\ 1 & -1 \\ 0 & L \end{bmatrix}$$

$$F_2 = \frac{B\delta A}{\alpha L} \begin{bmatrix} -1 & -L & 1 & 0 \\ 1 & 0 & -1 & L \end{bmatrix}$$

These submatrices are readily gathered into element matrices, which may then be assembled into global matrices in the usual manner. In fact, the element nodal degrees of freedom may be ordered so as to facilitate assembly, as follows:

$$x = \begin{bmatrix} \dot{q}_1 & q_1 & \dot{q}_2 & q_2 & p_1 & \dot{q}_3 & q_3 & \dot{q}_4 & q_4 & p_2 \end{bmatrix}^T$$

The elemental equations may then be expressed as

$$A\dot{x} + Bx = 0$$

In this case, the element matrices are

$$A = \begin{bmatrix} \frac{156\rho AL}{420} & 0 & \frac{22\rho AL^2}{420} & 0 & 0 & \frac{54\rho AL}{420} & 0 & -\frac{13\rho AL^2}{420} & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 \\ \frac{22\rho AL^2}{420} & 0 & \frac{4\rho AL^3}{420} & 0 & 0 & \frac{13\rho AL^2}{420} & 0 & -\frac{3\rho AL^3}{420} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & 0 & \frac{AL}{3} & 0 & 0 & 0 & 0 & \frac{AL}{6} \\ \frac{54\rho AL}{420} & 0 & \frac{13\rho AL^2}{420} & 0 & 0 & \frac{156\rho AL}{420} & 0 & -\frac{22\rho AL^2}{420} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ -\frac{13\rho AL^2}{420} & 0 & -\frac{3\rho AL^3}{420} & 0 & 0 & -\frac{22\rho AL^2}{420} & 0 & \frac{4\rho AL^3}{420} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \frac{AL}{6} & 0 & 0 & 0 & 0 & \frac{AL}{3} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & -\frac{\delta I}{L} & 0 & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{\delta I}{L} \\ -\mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & \frac{6EI}{L^2} & 0 & \frac{4EI}{L} & -\delta I & 0 & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} & \mathbf{0} \\ 0 & 0 & -\mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & -\frac{B\delta A}{\alpha L} & 0 & -\frac{B\delta A}{\alpha} & \frac{BAL}{3} & 0 & \frac{B\delta A}{\alpha L} & 0 & \mathbf{0} & \frac{BAL}{6} \\ 0 & -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{\delta I}{L} & 0 & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & -\frac{\delta I}{L} \\ \mathbf{0} & 0 & 0 & 0 & 0 & -\mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \frac{6EI}{L^2} & 0 & \frac{2EI}{L} & \mathbf{0} & 0 & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} & \delta I \\ 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & -\mathbf{1} & 0 & 0 \\ 0 & \frac{B\delta A}{\alpha L} & 0 & \mathbf{0} & \frac{BAL}{6} & 0 & -\frac{B\delta A}{\alpha L} & 0 & \frac{B\delta A}{\alpha} & \frac{BAL}{3} \end{bmatrix}$$

Again, note that E is the *unrelaxed* material modulus.

Numerical Results for Free Vibration Eigenvalue Problem

To evaluate the performance of this formulation of an ATF-damped beam element, a specific problem is addressed, namely the determination of the natural modes of vibration of a cantilevered beam. The geometric boundary conditions on displacement and rotation are explicitly enforced at $x = 0$.

The numerical parameter values used in this problem are

$E = 1$
$\rho = 1$
$I = 1$
$A = 1$
$L = 1.875$
$B = 34.39$
$\alpha = B$
$\delta = 2.623$

These values were chosen to give a fundamental frequency for an undamped beam of 1 and, using the results of Fourier analysis, to yield a peak damping ratio ζ of 0.05 near the frequency of the 4th mode. Again, note that α and δ cannot be uniquely specified.

The problem was studied using all combinations of alternate B and F coupling submatrices.[18] All but one combination produce generally good results, especially at low frequencies. There is an apparent discontinuity in the results for all of the combinations when the mode number becomes greater than the number of elements. At frequencies higher than this break point, the B_2-F_2 combination produces slightly better results.

The break in the results of damping vs frequency is not a significant concern for several reasons. First of all, frequencies estimated using finite elements are themselves inaccurate at high mode numbers, depending on the number of elements used to model a beam. In addition, at higher mode numbers, the Bernoulli-Euler beam theory is inherently limited (i.e., it neglects shear deformation and rotatory inertia).

Convergence of the results is studied using the elements based on the B_2-F_2 combination. Figure 5 shows the numerical results vs the Fourier results for 10, 20, and 40 elements. The results generally agree well; note that the frequency of the break point increases with the number of elements. The peak damping value of 5% is accompanied by a variation in modulus of about 20% from low to high frequencies. The first-order Fourier analysis does not include this variation in modulus and its effect on resonant frequencies—this accounts for the small disagreement between the Fourier analysis and the finite element results.

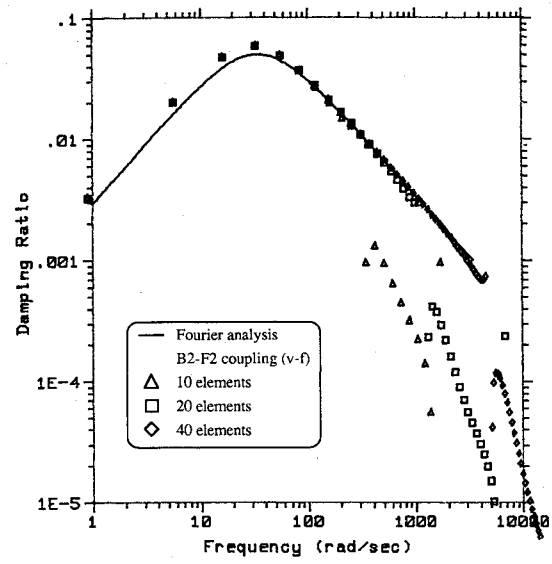

Fig. 5 Frequency-dependent damping exhibited in modal analysis of ATF-damped cantilevered beam.

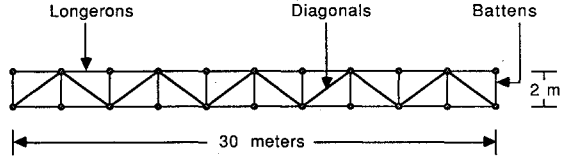

Fig. 6 A 10-bay planar truss structure.

Vibration of a Planar Space Truss

In this section, the ATF damping modeling method and the finite elements developed in the preceding sections are applied to the dynamic analysis of a planar truss structure. Such a structure is representative of those contemplated for use in future space systems.

This application requires the development of several aspects of the ATF approach, which were not explicitly addressed in the preceding sections. These include interelement continuity requirements for the augmenting thermodynamic fields and assembly of individual element matrices into global matrices (including coordinate transformations).

The Problem

Figure 6 illustrates the 10-bay planar truss structure used as the focus model for this development. It is an efficient beam-type structure built from three basic structural elements: longerons, which are parallel to the beam axis; diagonals, which bisect each rectangular bay; and battens, which are oriented transverse to the beam axis. The total length of the structure is 30 m, and the truss depth is 2 m.

The structure is assumed to behave as an ideal truss, i.e., the structural members carry only axial loads; no moments are transferred through the joints. Joint mass and other nonstructural masses are ignored. Finally, each kind of structural element is assumed to be made of a different material having different elastic, inertial, and dissipative properties. For this problem, it is assumed that a single augmenting thermodynamic field is sufficient to characterize the dissipative properties of each material in the frequency range of interest. Table 1 summarizes the relevant geometric and (isotropic) material properties of each kind of structural element.

Conventional damping analysis, if damping were considered at all, would likely employ the MSE method to estimate modal damping. This is an iterative process requiring analysis of the undamped structure to determine the frequencies and mode shapes of the undamped modes. The mode shapes are used to determine the distribution of strain energy over the structure, whereas the frequencies are used to determine frequency-dependent elastic and dissipative properties. Modal damping is then estimated for each mode and, roughly speaking, is numerically equal to the sum of material damping ratios weighted by the fraction of the strain energy stored in each material. The MSE method has serious errors when modes are closely spaced in frequency and when the damped mode shapes are much different than the undamped mode shapes.

Table 1 Properties of truss elements

	Longeron	Diagonal	Batten
Area, A (m^2)	31.$e-5$	19.$e-5$	6.3$e-5$
Unrelaxed modulus, E (Pa)	36.72e10	18.72e10	8.4e10
Density, ρ (kg/m^3)	2200.0	1600.0	2700.0
Peak damping, ζ_{peak}	0.005	0.01	0.05
Frequency of peak, $\omega(\zeta_{peak})$	200.0	2000.0	8000.0

As has been demonstrated, use of the ATF method generates modal damping in a single analysis, if that is all the analyst desires. However, it also yields more information: complex mode shapes and correspondingly more accurate results when undamped modes are closely spaced in frequency. Practical use of the ATF method, however, may require an initial analysis to establish the frequency range over which damping data is required; this can be used to limit the number of augmenting thermodynamic fields that must be introduced to model damping over the range of interest.

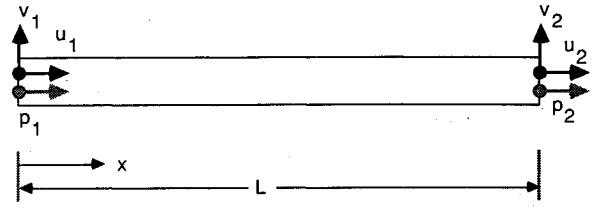

Fig. 7 An ATF-damped planar truss element.

Finite Elements for Truss Vibration

In a typical analysis of an ideal truss structure, a single rod-type element would be used to model an individual truss element. This is good practice because the kinds of motions of interest usually involve the entire structure—individual elements experience only simple extension and compression. Higher-order motions certainly are possible, i.e., a single element undergoing compression in one part, extension in another, but only at high mode numbers, where other assumptions may also be inappropriate.

When a single truss element is modeled using multiple rod-type finite elements, corresponding ATF are continuous from one element to the next. However, when different materials with different ATF meet at an interface, there is clearly no continuity, for each ATF represents different physical phenomena. A good rod-type finite element suitable for modeling truss behavior, therefore, will not require interelement continuity.

In numerical experiments without interelement ATF continuity, the linear-linear u-γ rod element discussed in a preceding section was observed to perform well.[18] The u-γ element is therefore suitable as a starting point for modeling truss structures and is used as the basis for what follows. This basic element must be modified to account for the kinetic energy associated with motion transverse to the axis of the element.

Figure 7 shows the truss element of interest along with the local element coordinate system. It has two degrees of freedom at each end (u and v) that correspond to physical displacements, one parallel to the rod axis (x) and one transverse to the rod axis (y). In addition, it has one degree of freedom at each end (p) to represent general displacements of the ATF, γ.

The governing matrix equations for a single element may be written generally in first-order form as

$$\begin{bmatrix} M_U & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & M_V & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & C \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \dot{u} \\ \ddot{v} \\ \dot{v} \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} 0 & K & 0 & 0 & B \\ -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \\ 0 & F & 0 & 0 & H \end{bmatrix} \begin{Bmatrix} \dot{u} \\ u \\ \dot{v} \\ v \\ p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where the submatrix M_U is identical to the submatrix M for the linear-linear u-γ element. In addition, if $v(x)$, the transverse mechanical displacement, is assumed to vary linearly over the element, the submatrix M_V is identical to M_U. All the other submatrices are identical to those for the linear-linear u-γ element. The details of the elements, not given here, may be deduced from the rod element presented earlier.

These element matrices may be assembled into global system matrices in the usual manner of structural finite-element analysis. This is a two-step process. The first step is to express each element matrix in terms of global coordinates rather than local, element coordinates. The second step is the actual assembly of global matrices.

Because this is a planar element, a single rotation, θ, different in general for each element, takes the global coordinates into the local element coordinates. The transformation from global to local coordinates may be written as

$$x_{\text{Local}} = Ux_{\text{Global}}$$

where x is the vector of discrete elemental coordinates. Note that no rotation need be applied to the ATF because it can be treated as a scalar field.

Element matrices expressed in terms of global coordinates are developed using a transformation that preserves the kinetic energy, strain energy, and dissipation power:

$$A_g = U^T A U \qquad B_g = U^T B U$$

Assembly of global system matrices is accomplished in the usual manner. Elemental terms corresponding to pairs of global degrees of freedom are added together to form the corresponding term in the global matrices. Fixed constraints are also enforced in the usual way, i.e., if a global displacement is constrained to be zero, corresponding rows and columns of the global matrices are eliminated from the problem.

In this application, assembly is performed in two steps. The first is the formation of a "superelement" corresponding to a two-bay section (repetitive element) of the truss by assembly of individual elements, and the second is the assembly of the superelements into the global system matrices. Each superelement has 42 first-order "states": 12 displacements, 12 velocities, and 18 ATF. When assembled into the global matrices, an additional batten is added.

The unconstrained first-order global matrices have dimensions of 170 by 170. As the free-free conditions of space are considered, no mechanical constraints are enforced.

Modal Analysis

In this section, an eigenvalue problem for the discrete matrix differential equations set up in the preceding section is identified and solved. Assuming characteristic motion of $e^{\lambda t}$, the global equations admit an eigenvalue problem:

$$[\lambda A_G + B_G]x = \mathbf{0}$$

This problem was formulated using the parameter values listed previously and solved using the EISPACK QZ routine. Figure 8 shows the results in terms of damping vs frequency. Where conventional analysis using the MSE method would have required considerable effort to generate modal damping values, the ATF method delivers them in a single, standard modal analysis. In addition, it delivers more accurate complex modes.

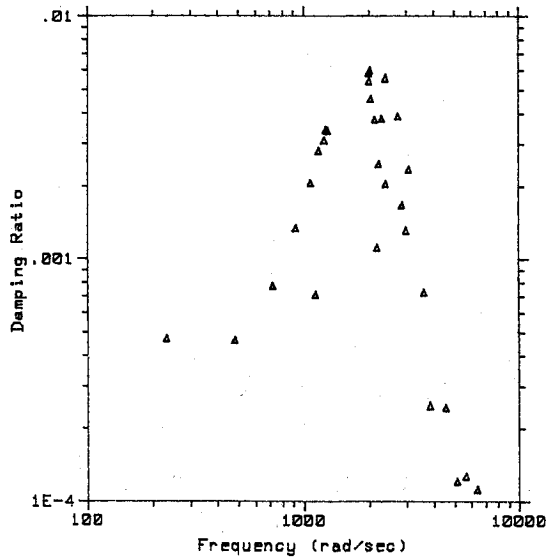

Fig. 8 Frequency-dependent damping exhibited in modal analysis of ATF-damped planar truss structure.

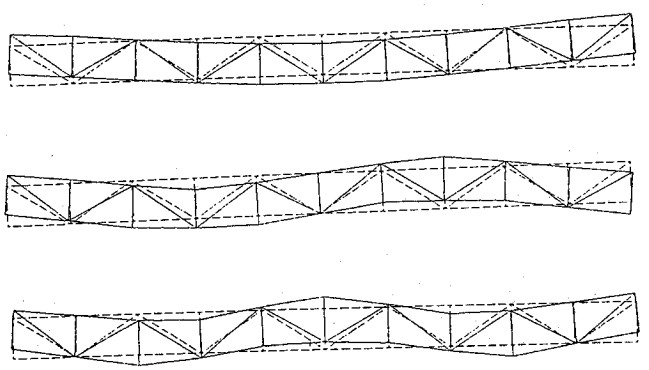

Fig. 9 Some mode shapes of the undamped planar truss structure.

Table 2 Comparison of undamped and ATF-damped truss modal analyses

Mode no.	Undamped ATF frequency, rad/s	Damped ATF frequency, rad/s	Damping radio, ζ, 10^{-3}
4	228.6	231.4	0.47
5	472.2	480.8	0.46
6	701.6	719.3	0.77
7	887.2	917.5	1.32
8	1024.0	1069.0	2.04
9	1102.0	1127.0	0.70
10	1109.0	1168.0	2.77

Note that the damping in the lowest bending modes increases quite rapidly with frequency. This is due to two unrelated factors: First, in this kind of truss structure, the fraction of strain energy stored in the diagonal members increases with mode number; this is analogous to the effects of transverse shear in an isotropic beam. Second, in this case, the damping of the material from which the diagonal members are made is increasing at frequencies below 2000 rad/s. Also note that the sixth flexible mode is an extensional mode and has considerably lower damping than neighboring bending modes.

For comparison, an undamped analysis was performed. Table 2 compares the results, and Fig. 9 depicts several of the low-frequency undamped mode shapes.

The ATF and undamped results agree well in terms of frequency, differing by about 1% for the first mode. The undamped frequencies are lower by as much as 5% in the 10th mode because the relaxed (static) modulus value was used for that analysis, as is usual practice. The ATF approach also yields modal damping. As has been demonstrated, the ATF method is compatible with, even a natural extension of, current structural finite element techniques.

Summary and Conclusions

A physically motivated material damping model compatible with current computational structural analysis methods has been developed. Termed the Augmenting Thermodynamic Fields method, its key feature is the introduction of additional fields to interact with the usual displacement field of continuum structural dynamics. Note that an increase in the accuracy of a structural dynamic model comes with a cost of dimensionality—additional coordinates are required to represent additional aspects of material behavior, viz, damping.

Coupled material constitutive equations and partial differential equations of evolution have been developed for microstructurally damped rods and for microstructurally damped planar bending beams. Damped finite elements have been developed and used to solve free vibration eigenvalue problems. When alternate finite elements existed for a kind of analysis, the best ones were identified. Numerical results compared favorably with results obtained using Fourier analysis. Damped rod elements were also the basis for a modal analysis of a large space truss structure.

Although a single augmenting field was generally discussed, the results are readily extended to multiple fields. In addition, although all the elements of the example structures were assumed to be damped, the method readily accommodates both damped and undamped elements. With the continued development of better analytical tools such as the subject method, damping will be modeled more accurately in the design of engineering systems and may ultimately become more accessible to design specification.

Acknowledgments

This work is based on the dissertation research of the first author,[18] which was performed under the guidance of the second. The first author wishes to thank the other members of his doctoral committee for their participatory support: Professors P. P. Friedmann, S. B. Dong, J. S. Gibson, and S. Putterman, as well as Dr. T. Nishimoto, who sparked the same author's interest in the general field of vibration damping. The financial support of SPARTA, Inc. through a tuition reimbursement program and free computer time is noted. Equally important was the flexibility allowed in setting a work schedule to meet the demands of part-time study. Thanks also to R. Quartararo and H. Rediess for their encouragement.

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